

# GENERALIZED FROLÍK CLASSES

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**ABSTRACT.** The class  $\mathfrak{C}$  relative to countably compact topological spaces and the class  $\mathfrak{P}$  relative to pseudocompact spaces introduced by Z. Frolík are naturally generalized relative to every topological property. We provide a characterization of such generalized Frolík classes in the broad case of properties defined in terms of filter convergence.

If a class of spaces can be defined in terms of filter convergence, then the same is true for its Frolík class.

Zdeněk Frolík [F1] introduced the class  $\mathfrak{C}$  of all topological spaces  $X$  such that  $X \times Y$  is countably compact, for every countably compact  $Y$ , and [F2] the class  $\mathfrak{P}$  of all topological spaces  $X$  such that  $X \times Y$  is pseudocompact, for every pseudocompact  $Y$ . Hence it is natural to define, for every class  $\mathcal{K}$  of topological spaces, the (*generalized*)  $\mathcal{K}$ -Frolík class  $\mathfrak{F}(\mathcal{K})$ , consisting of all topological spaces  $X$  such that  $X \times Y \in \mathcal{K}$ , whenever  $Y \in \mathcal{K}$ . If  $X \in \mathfrak{F}(\mathcal{K})$ , we shall sometimes simply say that  $X$  is  $\mathcal{K}$ -Frolík and, when convenient and with some sloppiness, we shall identify a topological property with the class of spaces satisfying it. Throughout the present note, “space” will be synonymous of “topological space”.

Notice that Frolík [F1, F2] assumed some separation axiom in the definitions of  $\mathfrak{C}$  and  $\mathfrak{P}$ ; however, in the present note, no separation axiom is necessary. Of course, every definition given here can be considered relative only to those spaces satisfying some separation axiom. If the separation axiom in question is preserved under products, then all theorems proved here remain valid. See [V1, V3] for results and variations on  $\mathfrak{C}$  and  $\mathfrak{P}$  without assuming separation axioms.

Classes analogous to  $\mathfrak{C}$  and  $\mathfrak{P}$  have subsequently been introduced by other authors. For example, in the above terminology, the famous and now solved Morita’s first conjecture asserts that  $\mathfrak{F}(\text{normal})$  is the class of all discrete spaces. See Atsuji [A] and Balogh [B] for more

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details and further references. Another class that is largely studied is the class of *productively Lindelöf spaces*,  $\mathfrak{F}(\text{Lindelöf})$  in the present terminology. See, e. g., Burton and Tall [BT]. Notice that the “class of spaces considered by Frolík” discussed in Burton and Tall’s paper refers to still another class of spaces introduced by Frolík, seemingly unrelated to the classes  $\mathfrak{C}$  and  $\mathfrak{P}$ .

It is useful to introduce also a relative notion. If  $\mathcal{K}$  and  $\mathcal{H}$  are classes of topological spaces, let us say that  $X$  is  *$\mathcal{K}$ -Frolík for  $\mathcal{H}$ -spaces* if  $X \times Y \in \mathcal{K}$ , whenever  $Y \in \mathcal{H}$ . In particular,  $\mathcal{K}$ -Frolík is the same as  $\mathcal{K}$ -Frolík for  $\mathcal{K}$ -spaces. Just to exemplify, the second now solved Morita conjecture, in the above terminology, asserts that metrizable spaces are exactly the normal-Frolík spaces for normal  $P$ -spaces.

We now turn to the kind of topological properties we shall consider here in connection with generalized Frolík classes. Notions related to ultrafilter convergence have frequently played an important role in the study of products. See the surveys García-Ferreira and Kocinac [GFK], Stephenson [St] and Vaughan [V2] for details and further references.

In [L1] we showed that if we extend some definitions by considering filters in place of ultrafilters we obtain genuinely more general notions. Hence we give here the general definitions; however, if the reader wants, he or she might always suppose that all the filters under consideration are ultrafilters, that is, maximal.

Recall that if  $X$  is a topological space and  $F$  is a filter over some set  $I$ , a sequence  $(x_i)_{i \in I}$  of elements of  $X$  is said to  *$F$ -converge* to  $x \in X$  if  $\{i \in I \mid x_i \in U\} \in F$ , for every neighborhood  $U$  of  $x$  in  $X$ .

**Definition 1.** [L1] Suppose that  $I$  is a set and  $\mathcal{F}$  is a family of filters over  $I$ . A topological space  $X$  is *sequencewise  $\mathcal{F}$ -compact* if, for every sequence  $(x_i)_{i \in I}$  of elements of  $X$ , there is  $F \in \mathcal{F}$  such that  $(x_i)_{i \in I}$   $F$ -converges in  $X$ . The class of all sequencewise  $\mathcal{F}$ -compact spaces shall be denoted by  $\mathcal{F}c$ . See [L1] and references there for some history about notions related to sequencewise  $\mathcal{F}$ -compactness.

If  $Y$  is a topological space and  $\mathbf{y} = (y_i)_{i \in I}$  is a sequence of elements of  $Y$ , the *spectrum of  $\mathbf{y}$  in  $Y$* , in symbols,  $\text{Spec}(\mathbf{y}, Y)$ , is the set of all filters  $F$  over  $I$  such that  $\mathbf{y}$   $F$ -converges in  $Y$ .

If  $\mathcal{H}$  is a class of topological spaces, the *spectrum of  $\mathcal{H}$  (relative to  $I$ )* is the set  $\mathbf{Spec}_I(\mathcal{H}) = \{\text{Spec}(\mathbf{y}, Y) \mid Y \in \mathcal{H} \text{ and } \mathbf{y} \text{ is an } I\text{-indexed sequence of elements of } Y\}$ .

**Proposition 2.** Suppose that  $\mathcal{F}$  is a set of filters over  $I$ .

A topological space  $X$  is  *$\mathcal{F}c$ -Frolík* if and only if  $X$  is sequencewise  $\mathcal{F} \cap \mathcal{G}$ -compact, for every  $\mathcal{G} \in \mathbf{Spec}_I(\mathcal{F}c)$ .

More generally, if  $\mathcal{H}$  is a class of topological spaces, then a topological space  $X$  is  $\mathcal{F}c$ -Frolík for  $\mathcal{H}$ -spaces if and only if  $X$  is sequencewise  $\mathcal{F} \cap \mathcal{G}$ -compact, for every  $\mathcal{G} \in \mathbf{Spec}_I(\mathcal{H})$ .

*Proof.* The first statement is the particular case of the second statement when  $\mathcal{H} = \mathcal{F}c$ . To prove the second statement, note the following chain of equivalences.

- (1)  $X$  is  $\mathcal{F}c$ -Frolík for  $\mathcal{H}$ -spaces;
- (2)  $X \times Y$  is sequencewise  $\mathcal{F}$ -compact, for every  $Y \in \mathcal{H}$ ;
- (3) for every  $Y \in \mathcal{H}$  and every sequence  $(z_i)_{i \in I}$  in  $X \times Y$ , there is  $F \in \mathcal{F}$  such that  $(z_i)_{i \in I}$   $F$ -converges in  $X \times Y$ ;
- (4) for every  $Y \in \mathcal{H}$  and every pair of sequences  $(y_i)_{i \in I}$  in  $Y$  and  $(x_i)_{i \in I}$  in  $X$ , there is  $F \in \mathcal{F}$  such that both  $(y_i)_{i \in I}$   $F$ -converges in  $Y$  and  $(x_i)_{i \in I}$   $F$ -converges in  $X$ ;
- (5) for every  $Y \in \mathcal{H}$  and every pair of sequences  $\mathbf{y} = (y_i)_{i \in I}$  in  $Y$  and  $(x_i)_{i \in I}$  in  $X$ , there is  $F \in \mathcal{F} \cap \text{Spec}(\mathbf{y}, Y)$  such that  $(x_i)_{i \in I}$   $F$ -converges in  $X$ ;
- (6)  $X$  is sequencewise  $\mathcal{F} \cap \text{Spec}(\mathbf{y}, Y)$ -compact, for every  $Y \in \mathcal{H}$  and every sequence  $\mathbf{y} = (y_i)_{i \in I}$  of elements of  $Y$ .
- (7)  $X$  is sequencewise  $\mathcal{F} \cap \mathcal{G}$ -compact, for every  $\mathcal{G} \in \mathbf{Spec}_I(\mathcal{H})$ .

The equivalence of (1) and (2) is the definition of  $\mathcal{F}c$ -Frolíkness for  $\mathcal{H}$  spaces; (2)  $\Leftrightarrow$  (3) follows from the definition of sequencewise  $\mathcal{F}$ -compactness; (3)  $\Leftrightarrow$  (4) follows from the easy and well-known fact that a sequence in a product  $F$ -converges if and only if each component of the sequence onto any factor  $F$ -converges; (4)  $\Leftrightarrow$  (5) is from the definition of  $\text{Spec}(\mathbf{y}, Y)$ ; (5)  $\Leftrightarrow$  (6) follows from the definition of sequencewise  $\mathcal{Q}$ -compactness, for  $\mathcal{Q} = \mathcal{F} \cap \text{Spec}(\mathbf{y}, Y)$ ; finally, (6)  $\Leftrightarrow$  (7) follows from the definition of  $\mathbf{Spec}_I(\mathcal{H})$ .  $\square$

Proposition 2 shows that the study of  $\mathcal{F}c$ -Frolíkness can be divided into two steps. In the first step one has to determine  $\mathbf{Spec}_I(\mathcal{F}c)$ , that is, all possible values for  $\text{Spec}(\mathbf{y}, Y)$ ,  $Y$  varying in  $\mathcal{F}c$ , and  $\mathbf{y}$  varying among all  $I$ -indexed sequences on  $Y$ . In the second step one should characterize those spaces which are sequencewise  $\mathcal{Q}$ -compact for all  $\mathcal{Q}$  having the form  $\mathcal{F} \cap \mathcal{G}$ , for some  $\mathcal{G} \in \mathbf{Spec}_I(\mathcal{F}c)$ .

Each of the two steps might prove to be very difficult.

Concerning the first step, one could notice that there are very little constraints on the values that  $\text{Spec}(\mathbf{y}, Y)$  might assume, in general. For simplicity, suppose that  $\mathcal{G}$  is a set of ultrafilters over  $I$ , and that  $\mathcal{G}$  contains all principal ultrafilters. Give  $I$  the discrete topology, let  $\beta(I)$  be its Čech-Stone compactification and, as usual, identify  $I$  with the set of all principal ultrafilters over  $I$ . Thus,  $I \subseteq \mathcal{G} \subseteq \beta(I)$ . Think

of  $\mathcal{G}$  as a subspace  $Y$  of  $\beta(I)$ , with the induced topology. Letting  $\mathbf{y}$  be the sequence which is the identity on  $I$ , it is trivial to check that  $\text{Spec}(\mathbf{y}, Y) = \mathcal{G} \cup \{P(I)\}$ . Thus  $\text{Spec}(\mathbf{y}, Y)$  can be quite arbitrary (of course, it necessarily contains all the principal ultrafilters, as well as the improper filter). On the other hand, elaborated constraints arise if we want  $Y$  to be sequencewise  $\mathcal{F}$ -compact, for some  $\mathcal{F}$ , since, in order to fulfill this request, we need to check that *for every* sequence, there is  $F \in \mathcal{F}$  such that the sequence  $F$ -converges.

As far as the second step is concerned, the difficulties of studying sequencewise  $\mathcal{F}$ -compactness have been hinted in [L1, Section 6].

In any case, Proposition 2 shows that the study of  $\mathcal{F}c$ -Frolíkness can be reduced to the study of simultaneous sequencewise  $\mathcal{Q}$ -compactness, for a (perhaps large number of) appropriate sets  $\mathcal{Q}$ . The above observation suggests the following definition.

**Definition 3.** Suppose that  $\mathbf{F} = \{\mathcal{F}_a \mid a \in A\}$ , where each  $\mathcal{F}_a$  is a family of filters over some set  $I_a$ . We say that a topological space  $X$  is **F-compact** if  $X$  is sequencewise  $\mathcal{F}_a$ -compact, for every  $a \in A$ .

The class of all **F-compact** spaces shall be denoted by  $\mathbf{F}c$ .

**F-compactness** is not necessarily equivalent to sequencewise  $\mathcal{F}$ -compactness, for some  $\mathcal{F}$ . See [L2].

Let us denote by  $(\mathcal{K} : \mathcal{H})$  the class of all spaces which are  $\mathcal{K}$ -Frolík for  $\mathcal{H}$ -spaces.

**Theorem 4.** For every  $\mathbf{F} = \{\mathcal{F}_a \mid a \in A\}$  as above, there is  $\mathbf{Q} = \{\mathcal{Q}_b \mid b \in B\}$  such that  $\mathfrak{F}(\mathbf{F}c) = \mathbf{Q}c$ .

More generally, for every  $\mathbf{F} = \{\mathcal{F}_a \mid a \in A\}$  and every class  $\mathcal{H}$  of topological spaces, there is  $\mathbf{Q} = \{\mathcal{Q}_b \mid b \in B\}$  such that  $(\mathbf{F}c : \mathcal{H}) = \mathbf{Q}c$ .

*Proof.* The first statement follows from the second one, by taking  $\mathcal{H} = \mathbf{F}c$ , since, by the definitions,  $(\mathbf{F}c : \mathbf{F}c) = \mathfrak{F}(\mathbf{F}c)$ .

Let us prove the rest of the theorem. Since, by definition,  $\mathbf{F}c = \bigcap_{a \in A} \mathcal{F}_a c$ , we have that  $(\mathbf{F}c : \mathcal{H}) = \bigcap_{a \in A} (\mathcal{F}_a c : \mathcal{H})$ , by an obvious property of the binary operator  $(- : -)$ . By the last statement in Proposition 2, for every  $a \in A$ , there are a set  $B_a$  and families  $(\mathcal{Q}_b)_{b \in B_a}$  such that a topological space  $X$  belongs to  $(\mathcal{F}_a c : \mathcal{H})$  if and only if  $X$  is sequencewise  $\mathcal{Q}_b$ -compact, for every  $b \in B_a$ . Since  $(\mathbf{F}c : \mathcal{H}) = \bigcap_{a \in A} (\mathcal{F}_a c : \mathcal{H})$ , we get that a topological space  $X$  belongs to  $(\mathbf{F}c : \mathcal{H})$  if and only if  $X$  is  $\mathbf{Q}$ -compact, for  $\mathbf{Q} = \{\mathcal{Q}_b \mid b \in \bigcup_{a \in A} B_a\}$ .  $\square$

The proofs of Proposition 2 and of Theorem 4 give an explicit description of the  $\mathbf{Q}$  given by 4.

**Corollary 5.** *For every class  $\mathcal{H}$  of topological spaces and every  $\mathbf{F}$  as in Definition 3,  $(\mathbf{F}c : \mathcal{H}) = \mathbf{Q}c$ , where  $\mathbf{Q} = \{\mathcal{F} \cap \mathcal{G} \mid \mathcal{F} \in \mathbf{F}, \mathcal{G} \in \text{Spec}_I(\mathcal{H})\}$ .*

*In particular, the  $\mathbf{Q}$  given by Theorem 4 can be chosen in such a way that, for every  $\mathcal{Q} \in \mathbf{Q}$ , there is  $\mathcal{F} \in \mathbf{F}$  such that  $\mathcal{Q} \subseteq \mathcal{F}$ . If this is the case, and every  $\mathcal{F} \in \mathbf{F}$  is a family of ultrafilters, then every  $\mathcal{Q} \in \mathbf{Q}$  is a family of ultrafilters.*

**Corollary 6.** (a) *There is  $\mathbf{Q}$  such that  $\mathfrak{F}(\text{Lindelöf}) = \mathbf{Q}c$ .*

(b) *There is  $\mathbf{Q}$  such that  $\mathfrak{F}(\text{linearly Lindelöf}) = \mathbf{Q}c$ .*

*Proof.* (a) The Lindelöf property can be expressed as the conjunction of  $[\omega_1, \lambda]$ -compactness, for every cardinal  $\lambda > \omega$ , equivalently, as the conjunction of  $[\lambda, \lambda]$ -compactness for every cardinal  $\lambda > \omega$ . Recall that a space is  $[\mu, \lambda]$ -compact if every open cover by at most  $\lambda$  sets has a subcover with  $< \mu$  sets.

Ginsburg and Saks [GS], Saks [Sa] and, in full generality, Caicedo [C] showed that  $[\mu, \lambda]$ -compactness is equivalent, in the present terminology, to sequencewise  $\mathcal{F}$ -compactness, for an appropriate  $\mathcal{F}$ . The corollary is now immediate from Theorem 4.

(b) A space is linearly Lindelöf if and only if it is  $[\lambda, \lambda]$ -compact, for every *regular* cardinal  $\lambda > \omega$ . This can be taken as the definition of linear Lindelöfness. The same arguments as in (a) give the result.  $\square$

The families given by Corollary 6 might possibly be proper classes. This can be dealt with the usual methods and causes no set-theoretical problem. Notice that Corollary 5 shows that, in both cases, the families  $\mathbf{Q}$  given by Corollary 6 can be chosen to consist only of ultrafilters.

It is probably interesting, and perhaps useful, to exemplify Proposition 2 and Theorem 4 in other particular cases. We have not yet worked out any detail.

The above arguments carry over without essential modifications in order to deal with pseudocompact-like notions. Just take into account everywhere sequences of nonempty open sets and their  $F$ -limit points, in place of sequences of elements and  $F$ -convergence.

**Definition 7.** Again, suppose that  $X$  is a topological space,  $F$  is a filter over some set  $I$  and  $\mathcal{F}$  is a family of filters over  $I$ .

A point  $x$  of  $X$  is said to be an  $F$ -limit point of a sequence  $(X_i)_{i \in I}$  of subsets of  $X$  if  $\{i \in I \mid X_i \cap U \neq \emptyset\} \in D$ , for every neighborhood  $U$  of  $x$  in  $X$ .

The topological space  $X$  is *sequencewise  $\mathcal{F}$ -pseudocompact* [L1] if for every sequence  $(O_i)_{i \in I}$  of nonempty open subsets of  $X$  there is  $F \in \mathcal{F}$  such that  $(O_i)_{i \in I}$  has an  $F$ -limit point in  $X$ . See again [GFK, L1] for

related notions and further references. The class of all sequencewise  $\mathcal{F}$ -pseudocompact spaces shall be denoted by  $\mathcal{F}p$ .

If  $Y$  is a topological space and  $\mathbf{O} = (O_i)_{i \in I}$  is a sequence of subsets of  $Y$ , the *pseudospectrum of  $\mathbf{O}$  in  $Y$* , in symbols,  $PSpec(\mathbf{O}, Y)$ , is the set of all filters  $F$  over  $I$  such that  $\mathbf{O}$  has an  $F$ -limit point in  $Y$ .

If  $\mathcal{H}$  is a class of topological spaces, the *pseudospectrum of  $\mathcal{H}$  (relative to  $I$ )* is the set  $\mathbf{PSpec}_I(\mathcal{H}) = \{PSpec(\mathbf{O}, Y) \mid Y \in \mathcal{H} \text{ and } \mathbf{O} \text{ is an } I\text{-indexed sequence of nonempty open subsets of } Y\}$ .

If  $\mathbf{F} = \{\mathcal{F}_a \mid a \in A\}$  and each  $\mathcal{F}_a$  is a family of filters over some set  $I_a$ , we say that a topological space  $X$  is **F**-pseudocompact if  $X$  is sequencewise  $\mathcal{F}_a$ -pseudocompact, for every  $a \in A$ .

The class of all **F**-pseudocompact spaces shall be denoted by  $\mathbf{F}p$ .

**Theorem 8.** *A space  $X$  is  $\mathcal{F}p$ -Frolík for  $\mathcal{H}$ -spaces if and only if  $X$  is sequencewise  $\mathcal{F} \cap \mathcal{G}$ -pseudocompact, for every  $\mathcal{G} \in \mathbf{PSpec}_I \mathcal{H}$ .*

*More generally, for every class  $\mathcal{H}$  of topological spaces and every  $\mathbf{F}$  as in Definition 3,  $(\mathbf{F}p : \mathcal{H}) = \mathbf{Q}p$ , where  $\mathbf{Q} = \{\mathcal{F} \cap \mathcal{G} \mid \mathcal{F} \in \mathbf{F}, \mathcal{G} \in \mathbf{PSpec}_I(\mathcal{H})\}$ .*

One needs very little structure, in order to define generalized Frolík classes. In fact, a binary operation suffices and, in the case of semi-groups, the Frolík operator satisfies some reasonable properties.

**Definition 9.** Suppose that  $\mathcal{C}$  is a class and a binary operation  $\times$  is defined on  $\mathcal{C}$ . If  $\mathcal{K}, \mathcal{H} \subseteq \mathcal{C}$ , we let  $(\mathcal{K} : \mathcal{H}) = \{X \in \mathcal{C} \mid X \times Y \in \mathcal{K}, \text{ for every } Y \in \mathcal{H}\}$ . This is sometimes called the *residuation operation*. We let  $\mathfrak{F}(\mathcal{K}) = (\mathcal{K} : \mathcal{K})$ .

Explicit reference to  $\mathcal{C}$  and  $\times$  shall not be indicated; however, as we shall show, the above notions are highly dependent on the choice of  $\mathcal{C}$ .

The next proposition lists some very easy properties of the operator  $\mathfrak{F}$ . Let us say that a class  $\mathcal{K} \subseteq \mathcal{C}$  is *factor closed* if, for every  $X, Y \in \mathcal{C}$ , we have that  $X \times Y \in \mathcal{K}$  implies  $X \in \mathcal{K}$ . In the particular case of topological spaces, if  $\mathcal{K}$  is closed under surjective images (and does not contain the empty space), then  $\mathcal{K}$  is factor closed.

**Proposition 10.** *Under the assumptions and notations in Definition 9, the following hold.*

- (a)  $\mathfrak{F}(\mathcal{K}) \supseteq \mathcal{K}$  if and only if  $\mathcal{K}$  is closed under  $\times$ .
- (b) If  $\mathcal{K} \neq \emptyset$  and  $\mathcal{K}$  is factor closed, then  $\mathfrak{F}(\mathcal{K}) \subseteq \mathcal{K}$ .
- (c) If  $\times$  is associative, then  $\mathfrak{F}(\mathcal{K})$  is closed under  $\times$  and  $\mathfrak{F}(\mathfrak{F}(\mathcal{K})) \supseteq \mathfrak{F}(\mathcal{K})$ .
- (d) If  $\mathcal{C}$  has an identity element, then  $\mathfrak{F}(\mathfrak{F}(\mathcal{K})) \subseteq \mathfrak{F}(\mathcal{K})$ .

*Proof.* We leave (a) and (b) to the reader.

(c) It is trivial to see that if  $\times$  is associative, then  $\mathfrak{F}(\mathcal{K})$  is closed under  $\times$ , hence (a) with  $\mathfrak{F}(\mathcal{K})$  in place of  $\mathcal{K}$  gives  $\mathfrak{F}(\mathfrak{F}(\mathcal{K})) \supseteq \mathfrak{F}(\mathcal{K})$ .

(d) Trivially the identity element  $E$  belongs to  $\mathfrak{F}(\mathcal{K})$ . Hence if  $Z \in \mathfrak{F}(\mathfrak{F}(\mathcal{K}))$  then  $Z = Z \times E \in \mathfrak{F}(\mathcal{K})$ .  $\square$

Strictly speaking, the class of topological spaces has not an identity element with respect to  $\times$ . However, considering the equivalence classes modulo homeomorphisms, the one-element space is indeed an identity element. Moreover, the product operation is associative among equivalence classes. Thus if  $\mathcal{K}$  is a class of topological spaces and  $\mathcal{K}$  is closed under homeomorphisms, then from Proposition 10(c)(d) we get  $\mathfrak{F}(\mathfrak{F}(\mathcal{K})) = \mathfrak{F}(\mathcal{K})$ .

The operator  $\mathfrak{F}$  is highly dependent on the class  $\mathcal{C}$ . Take  $\mathcal{C}$  to be the class of all topological spaces with at least two elements and let  $\mathcal{K}$  be the class of those spaces which have cardinality in the set  $\{2, 2^5\} \cup \{2^{9+2h} \mid h \in \omega\}$ . Then, relative to  $\mathcal{C}$ , the classes, respectively,  $\mathfrak{F}(\mathcal{K})$ ,  $\mathfrak{F}(\mathfrak{F}(\mathcal{K}))$ , and  $\mathfrak{F}(\mathfrak{F}(\mathfrak{F}(\mathcal{K})))$  are the classes of spaces with cardinality in, respectively,  $\{2^4\} \cup \{2^{8+2h} \mid h \in \omega\}$ ,  $\{2^{4+2h} \mid h \in \omega\}$  and  $\{2^{2+2h} \mid h \in \omega\}$ . Thus  $\mathfrak{F}(\mathcal{K}) \subset \mathfrak{F}(\mathfrak{F}(\mathcal{K})) \subset \mathfrak{F}(\mathfrak{F}(\mathfrak{F}(\mathcal{K})))$ , where the inclusions are strict. On the other hand, relative to the class of all topological spaces, we have showed that  $\mathfrak{F}(\mathcal{K}) = \mathfrak{F}(\mathfrak{F}(\mathcal{K}))$ .

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